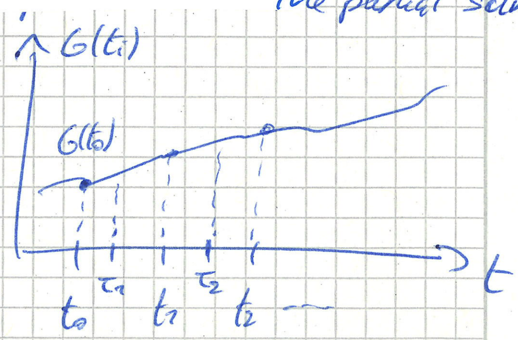
Ito's Calculus, Presentation

- Gaussian White Noise (it is a Markov process!)
- Wiener process: Properties, highlight on 3rd!
- Definition of the stochastic integral:

$G(t)$ an arbitrary function, $W(t)$ a Wiener process

The integral $\int_{t_0}^{t} G(t')dW(t') = \underset{n\to\infty}{\text{ms-limit}}\, S_n$ (mean square limit of the partial sum)

$$S_n = \sum_{i=1}^{n} G(\tau_i)(W(t_i) - W(t_{i-1}))$$



The sum (and so the integral) depends on the choice of τ_i!

Ito's Integral: $\tau_i = t_{i-1}$ so:

$$\int_{t_0}^{t} G(t')dW(t') = \underset{n\to\infty}{\text{ms-limit}} \sum_{i=1}^{n} G(t_{i-1})[W(t_i) - W(t_{i-1})]$$

Mean square limit: if $\lim_{n\to\infty} \langle (X_n - X)^2 \rangle = 0$ then $\underset{n\to\infty}{\text{ms-lim}}\, X_n = X$

Other definition of stochastic integral: Stratonovich Integral:

$\tau_i = t_{i-1}$ (like Ito) but $x_i \to \frac{1}{2}(x_i + x_{i-1})$ (with $x_i = x(t_i)$)

so $\int_{t_0}^{t} G(x(t'), t')dW(t') = \underset{n\to\infty}{\text{ms-lim}} \sum_{i=1}^{n} G(\frac{1}{2}(x_i + x_{i-1}), t_{i-1})[W(t_i) - W(t_{i-1})]$

Example: $G(x(t), t) = x(t)$

Ito: $(I)\int_{t_0}^{t} x(t')dW(t') = \underset{n\to\infty}{\text{ms-lim}} \sum_{i=1}^{n} x(t_{i-1})(W(t_i) - W(t_{i-1}))$

Stratonovich: $(S)\int_{t_0}^{t} x(t')dW(t') = \underset{n\to\infty}{\text{ms-lim}} \sum_{i=1}^{n} \frac{x(t_i) + x(t_{i-1})}{2}(W(t_i) - W(t_{i-1}))$

- Properties of Ito's Integral

Taylor expansion: $df(W(t), t) = \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}dt^2 + \frac{\partial f}{\partial W}dW + \frac{1}{2}\frac{\partial^2 f}{\partial W^2}dW^2 + \frac{\partial^2 f}{\partial W \partial t}dt\,dW + \ldots$

$$= \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial W^2}\right)dt + \frac{\partial f}{\partial W}dW$$

Proof of $dW(t)^2 = dt$ (We already saw $\langle dW^2 \rangle = dt$)

Ito's integral: $\int_{t_0}^{t} G(t')(dW(t'))^2 = \underset{n\to\infty}{\text{ms-limit}} \sum_{i=1}^{n} G_{i-1}(\underbrace{W_i - W_{i-1}}_{=\Delta W_i})^2 \overset{!}{=} \int_{t_0}^{t} G(t')\,dt'$

Define $I = \lim_{n\to\infty} \langle [\sum_i G_{i-1}(\Delta W_i^2 - \Delta t_i)]^2 \rangle$ $\Delta t_i = t_i - t_{i-1}$, $\sum_{i=1}^{n} \Delta t_i = t - t_0$

Calculate $\langle [\sum_i (\Delta W_i^2 - \Delta t_i)]^2 \rangle$

$= \langle [\sum_i \Delta W_i^2 - (t-t_0)]^2 \rangle = \langle \underbrace{\sum_i \Delta W_i^4 + 2\sum_{i<j} \Delta W_i^2 \Delta W_j^2}_{\text{use of } \sum_{i,j} = \sum_{i=j} + 2\sum_{i<j}}$

$- 2(t-t_0)\sum_i \Delta W_i^2 + (t-t_0)^2 \rangle$

use 3rd properties of Wiener process: $\Delta W_i \sim N(0, t_i - t_{i-1})$

so $\langle \sum_i \Delta W_i^4 \rangle = \sum_i \langle \Delta W_i^4 \rangle = 3\sum_i (t_i - t_{i-1})^2$

$\langle \Delta W_i^2 \Delta W_j^2 \rangle = \langle \Delta W_i^2 \rangle \langle \Delta W_j^2 \rangle = (t_i - t_{i-1})(t_j - t_{j-1})$

ΔW_i & ΔW_j are indep as $i \neq j$

$\langle [\sum_i \Delta W_i^2 - (t-t_0)]^2 \rangle = 3\sum_i (t_i - t_{i-1})^2 + 2\sum_{i<j} (t_i - t_{i-1})(t_j - t_{j-1}) - 2(t-t_0)\sum_i (t_i - t_{i-1}) + (t-t_0)^2$

$= 3\sum_i (t_i - t_{i-1})^2 + \sum_{i,j} (t_i - t_{i-1})(t_j - t_{j-1}) - \sum_i (t_i - t_{i-1})^2 - 2(t-t_0)\sum_i (t_i - t_{i-1}) + (t-t_0)^2$

$= 2\sum_i (t_i - t_{i-1})^2 + \sum_{i,j} [(t_i - t_{i-1}) - (t-t_0)][(t_j - t_{j-1}) - (t-t_0)]$ / $\sum_i (t_i - t_{i-1}) = t - t_0$

$= 2\sum_i (t_i - t_{i-1})^2 = 2\sum_i \Delta t_i^2 \longrightarrow 0$ as $n \to \infty$ (2nd order)

so $I = \lim_{n\to\infty} 2(\sum_i \Delta t_i^2 \langle G_{i-1}^2 \rangle) = 0$ (G_i has to be bounded)

$\langle [\sum_i G_{i-1}(\Delta W_i^2 - \Delta t_i)]^2 \rangle = 0$ means $\underset{n\to\infty}{\text{ms-lim}} (\sum_i G_{i-1}\Delta W_i^2 - \sum_i G_{i-1}\Delta t_i) = 0$

Finally, using: $\underset{n\to\infty}{\text{ms-lim}} \sum_i G_{i-1}\Delta W_i^2 = \int_{t_0}^{t} G(t')\,dW(t')^2$ and

$\underset{n\to\infty}{\text{ms-lim}} \sum_i G_{i-1}\Delta t_i = \int_{t_0}^{t} G(t')\,dt'$

one has: $\int_{t_0}^{t} G(t')\,dW(t')^2 = \int_{t_0}^{t} G(t')\,dt'$ so $dW(t)^2 = dt$ □.

Proof of $(dW(t))^{2+N}$, $N>0$ is similar but with higher moments of ΔW_i

Example: $\int_{t_0}^{t} W(t')\,dW(t') = \underset{n\to\infty}{\text{ms-lim}} \sum_i W_{i-1}\Delta W_i$

$= \frac{1}{2} \underset{n\to\infty}{\text{ms-lim}} \sum_i [(\underbrace{W_{i-1} + \Delta W_i}_{W_i})^2 - (W_{i-1})^2 - (\Delta W_i)^2] = \frac{1}{2} \underset{n\to\infty}{\text{ms-lim}} \sum_i [W_i^2 - W_{i-1}^2 - \Delta W_i^2]$

$= \frac{1}{2} \underset{n\to\infty}{\text{ms-lim}} (W(t)^2 - W(t_0)^2 - \sum_i \Delta W_i^2) = \frac{1}{2}(W(t)^2 - W(t_0)^2 - (t-t_0))$

$\underset{n\to\infty}{\text{ms-lim}} \sum_i \Delta W_i^2 = \underset{n\to\infty}{\text{ms-lim}} \sum_i \Delta t_i = t - t_0$

Ito's Calculus, Presentation (next)

- Taylor expansion of $df(x(t))$:

$$dx(t) = A(x(t),t)dt + \sqrt{D(x(t),t)}\, dW(t)$$

$$(dx(t))^2 = A^2(x(t),t)dt^2 + D(x(t),t)\,dW(t)^2 + 2A(x(t),t)\sqrt{D(x(t),t)}\,dt\,dW(t)$$

$$= D(x(t),t)dt \quad \text{as } dt^2 = dW^4 = 0 = dt\,dW = dW^3$$

$$\& \ dW^2 = dt$$

$$(dx(t))^3 = A(x(t),t)D(x(t),t)\,dt^2 + D(x(t),t)^{\frac{3}{2}}\,dt\,dW = 0$$

so $(dx(t))^{2+N} = 0$, $N>0$

so
$$\boxed{df(x(t)) = \{A(x(t),t)\, f'(x(t)) + \tfrac{1}{2} D(x(t),t) f''(x(t))\}dt + \sqrt{D(x(t),t)}\, f'(x(t))\, dW(t)}$$

Ito calculus (next)

Summary of presentation

Start from Gaussian white noise process: (this a Markov process!) (can deduce Brownian motion & others...)

$$\frac{dX(t)}{dt} = \underbrace{A(X(t),t)}_{\text{Drift}} + \underbrace{\sqrt{D(X(t),t)}}_{\text{Diffusion}}\ \underbrace{\Gamma(t)}_{\text{White noise fct}}$$

$\langle \Gamma(t)\rangle = 0$ & $\langle \Gamma(t)\Gamma(t')\rangle = \delta(t-t')$

so $dX(t) = A(X(t),t)dt + \sqrt{D(X(t),t)}\underbrace{\Gamma(t)dt}_{\equiv dW(t)}$

$dW(t)$ is a Wiener increment

In solving we expect $\int_0^t \Gamma(t')dt' \equiv \int_0^t dW(t') = W(t)$ a Wiener process (or standard Brownian motion)

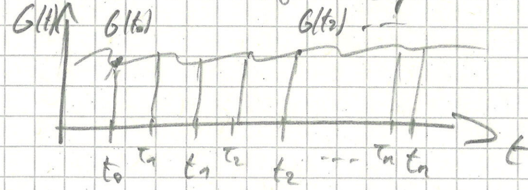
Properties of $W(t)$, a Wiener process: (we write $W_t \equiv W(t)$)

1. $W_0 = 0$
2. Independent increments: $\forall t > s \geq 0$, $W_t - W_s$ is independent of W_u lues
3. Normally distributed increments: $\forall t > s \geq 0$, $W_t - W_s \sim \mathcal{N}(0, t-s)$
4. W_t is continuous in t

Need to define the stochastic integral:

$G(t)$ an arbitrary function, $W(t)$ a Wiener process, the stochastic integral $\int_0^t G(t')dW(t')$ is defined as a limit of the partial sums:

$$S_n = \sum_{i=1}^n G(\tau_i)[W(t_i) - W(t_{i-1})]$$



↳ depend on the choice of τ_i!

Ito's Integral: $\tau_i = t_{i-1}$ so $\int_0^t G(t')dW(t') = \underset{n\to\infty}{\text{ms-lim}}\left(\sum_{i=1}^n G(t_{i-1})(W(t_i) - W(t_{i-1}))\right)$

Mean square limit: if $\lim_{n\to+\infty}\langle (X_n - X)^2\rangle = 0$ then $\underset{n\to+\infty}{\text{ms-lim}}\, X_n = X$

Properties of Ito's Integral: (main properties: relevant ones)

a) Existence whenever $G(t')$ is continuous & non-anticipating on $[0,t]$

b) General Differentiation Rule: for an arbitrary fct $f(W(t),t)$:

$$df(W(t),t) = \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}dt^2 + \frac{\partial f}{\partial W}dW + \frac{1}{2}\frac{\partial^2 f}{\partial W^2}dW^2 + \frac{\partial^2 f}{\partial W \partial t}dt\, dW + \ldots$$

$$= \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial W^2}\right)dt + \frac{\partial f}{\partial W}dW$$

Why all other terms goes to zero?

$\tau_i = t_{i-1}$ & new choice of τ_i

Stratonovich Integral: $\int_{t_0}^t G(x(t'),t')dW(t') = \underset{n\to\infty}{\text{ms-lim}}\sum_{i=1}^n G\left[\frac{1}{2}(x(t_i)+x(t_{i-1})), t_{i-1}\right](W(t_i) - W(t_{i-1}))$

Proof that $dW(t)^2 = dt$ & $(dW)^{2+N} = 0$ for $N>0$

From Ito's integral: $\int_{t_0}^{t} G(t')(dW(t'))^{2+N} \equiv \underset{n\to\infty}{\text{ms-lim}} \sum_i G_{i-1}(\underbrace{W_i - W_{i-1}}_{=\Delta W_i})^{2+N} = \begin{cases} 0 & \text{for } N>0 \\ \int_{t_0}^{t} G(t')dt', & N=0 \end{cases}$

Proof for $N=0$:

Define $I = \lim_{n\to+\infty} \langle [\sum_i G_{i-1}(\Delta W_i^2 - \Delta t_i)]^2 \rangle$ $\quad \Delta t_i = t_i - t_{i-1}$

and $\sum_{i=1}^{n} \Delta t_i = t_n - t_0 = t - t_0$

Calculate $\langle [\sum_i (\Delta W_i^2 - \Delta t_i)]^2 \rangle$ $\quad \sum_{i,j} = \sum_{i=j} + 2\sum_{i<j}$

$= \langle [\sum_i \Delta W_i^2 - (t-t_0)]^2 \rangle = \langle \sum_i \Delta W_i^4 + 2\sum_{i<j} \Delta W_i^2 \Delta W_j^2 - 2(t-t_0)\sum_i \Delta W_i^2 + (t-t_0)^2 \rangle$

we use the properties of Wiener process: $\Delta W_i \sim \mathcal{N}(0, t_i - t_{i-1})$

$\Delta W_i \Delta W_j$ are independent for $i \neq j$

$\to \langle \sum_i \Delta W_i^4 \rangle = \sum_i 3(t_i - t_{i-1})^2$

$\langle \Delta W_i^2 \Delta W_j^2 \rangle = (t_i - t_{i-1})(t_j - t_{j-1})$ $\quad$ use $2\sum_{i<j} = \sum_{i,j} - \sum_{i=j}$

so $\langle [\sum_i \Delta W_i^2 - (t-t_0)]^2 \rangle = \sum_i 3(t_i - t_{i-1})^2 + \sum_{i,j}(t_i - t_{i-1})(t_j - t_{j-1}) - \sum_i (t_i - t_{i-1})^2$

$- 2(t-t_0)\sum_i (t_i - t_{i-1}) + (t-t_0)^2$ $\quad$ use $\sum_i t_i - t_{i-1} = t - t_0$

$= 2\sum_i (t_i - t_{i-1})^2 + \sum_{i,j}[(t_i - t_{i-1}) - (t-t_0)][(t_j - t_{j-1}) - (t-t_0)]$

$= 2\sum_i (t_i - t_{i-1})^2 = 2\sum_i \Delta t_i^2$ $\quad \sum_{i=1}^{3}(t_i - t_{i-1})^2 = (t_1-t_0)^2 + (t_2-t_1)^2 + (t_3-t_2)^2$

$= t_1^2 + t_0^2 - 2t_1t_0 + t_2^2 + t_1^2 - 2t_1t_2$

so $I = \lim_{n\to\infty} 2(\sum_i \Delta t_i^2 \langle G_{i-1}^2 \rangle) = 0$

as $\lim_{n\to\infty} \sum_i \Delta t_i^2 = 0$ $\quad$ (G_i has to be bounded)

so $\lim_{n\to+\infty} \langle [\sum_i G_{i-1}(\Delta W_i^2 - \Delta t_i)]^2 \rangle = 0$ means $\underset{n\to\infty}{\text{ms-lim}}(\sum_i G_{i-1}\Delta W_i^2 - \sum_i G_{i-1}\Delta t_i) = 0$

Moreover, using Ito's integral: $\underset{n\to\infty}{\text{ms-lim}} \sum_i G_{i-1}\Delta W_i^2 = \int_{t_0}^{t} G(t')(dW(t'))^2$

and $\underset{n\to+\infty}{\text{ms-lim}} \sum_i G_{i-1}\Delta t_i = \int_{t_0}^{t} G(t')dt'$

so $(dW(t))^2 = dt$

for $(dW(t))^{2+N}$, $N>0$ use same method but with higher moments ($\Delta W_i \sim \mathcal{N}(0,\Delta t_i)$)

so $dW\,dt = (dW)^3 = 0$, $dt^2 = dW^4 = 0$

so $df(W(t),t) = (\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial W^2})dt + \frac{\partial f}{\partial W}dW$ is proven

Calculus of $\int_{t_0}^{t} W(t')dW(t')$ as an example

$\int_{t_0}^{t} W(t')dW(t') = \underset{n\to\infty}{\text{ms-lim}}(\sum_{i=1}^{n} W_{i-1}\Delta W_i) = \frac{1}{2}\underset{n\to\infty}{\text{ms-lim}}(\sum_{i=1}^{n}[(\overbrace{W_{i-1}+\Delta W_i}^{W_i})^2 - (W_{i-1})^2 - (\Delta W_i)^2])$

$= \frac{1}{2}\underset{n\to\infty}{\text{ms-lim}}(\sum_{i=1}^{n}[W_i^2 - W_{i-1}^2 - (\Delta W_i)^2]) = \frac{1}{2}\underset{n\to\infty}{\text{ms-lim}}(W(t)^2 - W(t_0)^2 - \sum_{i=1}^{n}(\Delta W_i)^2)$

from before: $\underset{n\to\infty}{\text{ms-lim}}(\sum_i^n (\Delta W_i)^2) = \underset{n\to\infty}{\text{ms-lim}}(\sum_i^n \Delta t_i) = t - t_0$

$= \frac{1}{2}[W(t)^2 - W(t_0)^2 - (t-t_0)]$

so $\int_{t_0}^{t} W(t')dW(t') = \frac{1}{2}[W(t)^2 - W(t_0)^2 - (t-t_0)]$

Ito calculus (next)

Summary of presentation (next)

Ito Stochastic Differential Equations (SDE)

$$dX(t) = A(X(t),t)\,dt + \sqrt{D(X(t),t)}\,dW(t)$$

Property: a) Existence (not proven...)
b) Uniqueness of solutions (not proven)

We can now look at an arbitrary function $f(x(t))$ and what Ito SDE it obeys:

$$df(x(t)) = f(x(t)+dx(t)) - f(x(t))$$
$$= \cancel{f(x(t))} + dx(t)\frac{df(x)}{dx} + \frac{1}{2}(dx(t))^2\frac{d^2f(x)}{dx^2} + \frac{1}{3}(dx(t))^3\frac{d^3f(x)}{dx^3} + \ldots - \cancel{f(x(t))}$$

$$dX(t) = A(X(t),t)\,dt + \sqrt{D(X(t),t)}\,dW(t)$$
$$(dX(t))^2 = A(X(t),t)^2 dt^2 + 2A(X(t),t)\sqrt{D(X(t),t)}\,dt\,dW(t) + D(X(t),t)(dW(t))^2$$
$$= D(X(t),t)\,dt \quad \text{as } dt^2 = 0 = dt\,dW \;\&\; dW^2 = dt$$
$$(dX(t))^3 = (dX(t))^2\,dX(t) = D(X(t),t)\,A(X(t),t)\,dt^2 + (D(X(t),t))^{\frac{3}{2}}\,dt\,dW = 0$$

$\vdots$

$$(dX(t))^{2+N} = 0 \quad , N>0$$

so
$$\boxed{df(X(t)) = \left\{A(X(t),t)\,f'(X(t)) + \frac{1}{2}D(X(t),t)\,f''(X(t))\right\}dt + \sqrt{D(X(t),t)}\,f'(X(t))\,dW(t)}$$

Ito's lemma